SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 4

MATH 241

17.8.1

Proof. Solve $\sin z = -i$, i.e. $\frac{e^{iz}-e^{-iz}}{2i} = -i$, so we have e^{iz} as the solution to quadratic equation $t^2 - 2t - 1 = 0$, so $e^{iz} = 1 \pm \sqrt{2}$, $iz = \ln(1 \pm \sqrt{2}) + 2n\pi i$, eventually we have

$$z = -i \ln(1 \pm \sqrt{2}) + 2n\pi, \quad n \in \mathbb{Z}$$

□

17.8.10

Proof. Solve $\tan z = 3i$, i.e., $\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} = 3i$. So $e^{iz} - e^{-iz} = -3(e^{iz} + e^{-iz})$, or equivalently $4e^{iz} = -2e^{-iz}$, so $e^{2iz} = -\frac{1}{2}$, $2iz = \ln(\frac{1}{2}) + i(\pi + 2n\pi)$, eventually we have

$$z = -\frac{i}{2}\ln\frac{1}{2} + \frac{\pi + 2n\pi}{2} = \frac{i}{2}\ln 2 + \frac{\pi + 2n\pi}{2} \ \ n \in \mathbb{Z}$$

□

18.1.5

Proof. The curve is parametrized as $z(t) = \cos t + i \sin t = e^{it}$, t from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $z'(t) = ie^{it}$, so the integral is

$$\int_0^{2\pi} \frac{1 + e^{it}}{e^{it}} ie^{it} dt = \int_0^{2\pi} (1 + e^{it}) dt = 2\pi$$

Here I omit some steps at the end, but you should already be familiar with the calculation of $\int_0^{2\pi} e^{it} dt$! □

18.1.14

Proof. We can parametrize the ellipse as $x = 6\cos t$, $y = 2\sin t$, t from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$. So $z(t) = 6\cos t + 2i \sin t$, $z'(t) = -6\sin t + 2i\cos t$. The integral is $\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (-6\sin t + 2i \cos t) dt = (6\cos t + 2i \sin t)|_{\frac{\pi}{2}}^{-\frac{\pi}{2}} = -2i - 2i = -4i$. □

18.1.29

Proof. (a) Now the integrand is the constant function 1, so

$$\int_C dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta z_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (z_k - z_{k-1}) = \lim_{\|P\|\to 0} z_n - z_0 = z_n - z_0$$

Since z_n and z_0 do not depend on the partition or choice of points at all.

(b) In problem 18.1.14, it is the integral of constant function 1 along the curve from $2i$ to $-2i$, so by the formula we derived in (a), the integral is just equal to the difference between the end point and the starting point, which is $-2i - 2i = -4i$. □

Fall 10, # 3

Proof. C is parametrized by $z(t) = \cos t + i \sin t$, t from 0 to 2π. So $\oint_C Re(z)dz = \int_0^{2\pi} \cos t(-\sin t + i\cos t)dt = \int_0^{2\pi} i\cos^2 t - \sin t \cos t dt$. By doubel angle formulae, $\cos^2 t = \frac{\cos 2t+1}{2}$, $\sin t \cos t = \frac{\sin 2t}{2}$, so we get

$$\int_0^{2\pi} i\frac{\cos 2t + 1}{2} - \frac{\sin 2t}{2} dt = \pi i$$

□